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ON QUASI-DUAL BAER MODULES

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Abstract. We introduce and study the notions of quasi-dual Baer modules, FI-$\mathcal{T}$-non-cosingular modules and FI-$\mathcal{K}$-modules. We show that a module M is a quasi-dual Baer and FI-$\mathcal{K}$-module if and only if it is FI-lifting and FI-$\mathcal{T}$-non-cosingular. A necessary condition for a direct sum of quasi-dual Baer modules to be quasi-dual Baer are obtained. A characterization is given of when a module is quasi-dual Baer, a necessary condition being that the endomorphism ring itself is a left quasi-Baer ring.

Keywords: quasi-dual Baer modules, quasi-Baer rings, FI-lifting modules, endomorphism rings.

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1. Introduction

Throughout this paper, R will denote an arbitrary associative ring with identity, M a unitary right R-module and $S = End_R(M)$ the ring of all R-endomorphisms of M. Kaplansky introduced the concept of Baer rings in 1955 [3] and Clark introduced the notion of quasi-Baer rings in 1967 [2]. A ring R is called left *Baer (quasi-Baer)* if the left annihilator of any nonempty subset (left ideal) of R is generated as a left ideal by an idempotent. Rizvi and Roman introduced the concepts of Baer and quasi-Baer modules in [5]. According to [5], M is called a *Baer* (respectively *quasi-Baer*) module if the right annihilator in M of any left ideal (respectively ideal) of S is a direct summand of M. In [10], Keskin-Tütüncü and Tribak dualized the concept of Baer modules. According to [10], a module M is called a *dual Baer* module if for every submodule N of M, there exists an idempotent e in S such that $\{\phi \in S \mid \operatorname{Im}\phi \subseteq N\} = eS$. In this work we introduce the notion of quasi-dual Baer modules. A module M is called a *quasi-dual Baer* module if for every fully invariant submodule N of M, there exists an idempotent e in S such that $\{\phi \in S \mid \operatorname{Im}\phi \subseteq N\} = eS$. Obviously, any dual Baer module is quasi-dual Baer.

We will use the notation $N \leq_e M$ to indicate that N is essential in M (i.e., $\forall 0 \neq L \leq M,\ N \cap L \neq 0$); $N \ll M$ means that N is small in M (i.e., $\forall L \lneq M,\ L + N \neq M$); $N \trianglelefteq M$ means that N is a fully invariant submodule of M (i.e., $\forall \phi \in End_R(M),\ \phi(N) \subseteq N$). The notation $N \leq^{\oplus} M$ denotes that N is a direct summand of M. For all $I \subseteq S$, the left and right annihilators of I in S are denoted by $\ell_S(I)$ and $r_S(I)$, respectively. We also denote $r_M(I) = \{x \in M \mid Ix = 0\}$, $E_M(I) = \sum_{\phi \in I} \operatorname{Im}\phi$, for $I \subseteq S$; $\ell_S(N) = \{\phi \in S \mid \phi(N) = 0\}$, $D_S(N) = \{\phi \in S \mid \operatorname{Im}\phi \subseteq N\}$, for $N \subseteq M$.

Recall that a module M is called a *lifting* module if, every submodule N of M can be written in the form $N = A \oplus D$ where A is a direct summand of M and $D \ll M$ [4]. A module M is

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called a *FI-lifting* module if, every fully invariant submodule N of M can be written in the form $N = A \oplus D$ where A is a direct summand of M and $D \ll M$.

In Section 2, we introduce and study the notions of quasi-dual Baer modules, FI-$\mathcal{T}$-noncosingular modules and FI-$\mathcal{K}$-modules. Close links of quasi-dual Baer modules to FI-lifting modules are established. We prove that an arbitrary direct sum of quasi-dual Baer modules that are isomorphic to a factor of each other is quasi-dual Baer. We show that any direct summand of a quasi-dual Baer module is quasi-dual Baer. We remark that some of our results obtained in this paper are dual to those obtained by [5].

In Section 3, we show that the endomorphism ring of a quasi-dual Baer module is a left quasi-Baer ring and obtain a necessary condition for the converse to hold true.

Lemma 1.1. *Let M be a module, and $M = M_1 \oplus M_2$ be a direct sum decomposition. If $N \trianglelefteq M$ then $N = N_1 \oplus N_2$ where $N_i = N \cap M_i \trianglelefteq M_i$, for $i = 1, 2$.*

Proof. See [5, Lemma 1.10]. □

Lemma 1.2. *Let M be a module, with $M = N_1 \oplus N_2$ and let $F_1 \trianglelefteq N_1$. Then there exists $F_2 \trianglelefteq N_2$, so that $F_1 \oplus F_2 \trianglelefteq M$.*

Proof. See [5, Lemma 1.11]. □

Lemma 1.3. *For $N \leq M$, $I \leq S$, $P \trianglelefteq M$, $L \trianglelefteq S$, the following hold:*

(*i*) $E_M(D_S(E_M(I))) = E_M(I)$

(*ii*) $D_S(E_M(D_S(N))) = D_S(N)$

(*iii*) $E_M(L) \trianglelefteq M$

(*iv*) $D_S(P) \trianglelefteq S$.

Proof. (*i*) $E_M(D_S(E_M(I))) = \sum_{\phi \in D_S(E_M(I))} \operatorname{Im}\phi \subseteq E_M(I)$. Conversely, since $I \subseteq D_S(E_M(I))$, then $E_M(I) \subseteq E_M(D_S(E_M(I)))$.

(*ii*) Similar to the proof of (*i*).

(*iii*) Let $L \trianglelefteq S$ and $f \in S = End_R(M)$, then $f(\sum_{\phi \in L} \operatorname{Im}\phi = \sum_{\phi \in L} \operatorname{Im} f\phi \leq \sum_{\phi \in L} \operatorname{Im}\phi$ (since $\phi \in L$ and $L \trianglelefteq S$, thus $f\phi \in L$). Therefore $E_M(L) \trianglelefteq M$.

(*iv*) We observe that, $D_S(P) \leq S_S$. On the other hands, if $\phi \in D_S(P)$, then $\forall \psi \in S$, $\psi\phi(M) \subseteq \psi(P) \subseteq P$ since $P \trianglelefteq M$. Hence $\psi\phi \in D_S(P)$. Therefore $D_S(P) \trianglelefteq S$. □

2. Quasi-dual Baer modules

We say that a module M is a *quasi-dual Baer* module if for every fully invariant submodule N of M, there exists an idempotent e in S such that $D_S(N) = eS$, or equivalently, for every ideal I of S, $E_M(I)$ is a direct summand of M. Any semisimple module is a quasi-dual Baer. Obviously, any dual Baer module is quasi-dual Baer.

Lemma 2.1. *Let M be a quasi-dual Baer module and $\phi \in S$. If $Im\phi \trianglelefteq M$, then $Im\phi \leq^{\oplus} M$.*

Proof. Let $\phi \in S$ such that $\operatorname{Im}\phi \trianglelefteq M$. We have $\operatorname{Im}\phi = E_M(S\phi S)$ since $\operatorname{Im}\phi \trianglelefteq M$. Thus $\operatorname{Im}\phi \leq^{\oplus} M$. □

The quasi-dual Baer property does not always transfer from a module to each of its submodules as the next example demonstrates.

Example 2.1. The $\mathbb{Z}$-module $\mathbb{Q}$ is quasi-dual Baer but the submodule $\mathbb{Z}$ is not a quasi-dual Baer $\mathbb{Z}$-module.

Next, we see that a direct summand of a quasi-dual Baer module inherits the property.

Theorem 2.1. *Every direct summand of a quasi-dual Baer module M is quasi-dual Baer.*

Proof. Let $N \leq^{\oplus} M$, then there exists $e^2 = e \in S$ such that $N = eM$. Assume that $F \trianglelefteq N$, then by Lemma 1.2, there exists $G \trianglelefteq (1-e)M$ such that $F \oplus G \trianglelefteq M$. Since M is quasi-dual Baer, $I = D_S(F \oplus G) \trianglelefteq^{\oplus} S$. As $End_R(N) = eSe$, and $I \trianglelefteq S$, $eIe = eSe \cap I$. We have $I = fS$ where $f^2 = f \in S$, and so $eIe = efSe$. But since $fS \trianglelefteq S$, $ef \in fS$. Hence $ef = fef$. We can write $eIe = efSe = fefSe = efefSe = (efe)(efSe)$. Notice that $(efe)^2 = efe$. We have $(efe)(efSe) \subseteq (efe)(eSe)$, but the reverse: let $(efe)(ese) \in (efe)(eSe)$, then $efeese = efese =$ $= fefese = efefese = (efe)(ef(es)e) \in (efe)(efSe)$. Hence we have that $eIe \leq^{\oplus} eSe$. Now we show that $eIe = D_{eSe}(F)$. We see that $eie(M) \subseteq ei(M) \subseteq e(F \oplus G) = eF + eG \subseteq F$, for $i \in I$, therefore $eIe \subseteq D_{eSe}(F)$. Assume that $0 \neq eje \in eSe$ such that $eje(M) \subseteq F$. Hence $eje(M) \subseteq F \oplus G$ and so $eje \in D_{eSe}(F \oplus G) = I$. But $eje = eejee = e(eje)e \in eIe$. Therefore $D_{eSe}(F) = eIe \leq^{\oplus} eSe$. F is arbitrary, hence N is quasi-dual Baer. $\square$

Recall that a module M is said to have the *FI-summand sum property* (FI-SSP) if the sum of two fully invariant direct summands is again a direct summand. A module M has the *FI-strong summand sum property* (FI-SSSP) if the sum of any number of fully invariant direct summands is again a direct summand.

Lemma 2.2. *Every quasi-dual Baer module M has the FI-strong summand sum property (FI-$SSSP$).*

Proof. Let $e_iM \trianglelefteq M$ where $e_i^2 = e_i \in S$, and $i \in \Lambda$ (Λ is an index set). Then $e_iS \trianglelefteq S$ ($i \in \Lambda$). Define $I = \sum_{i\in\Lambda} e_iS$, then $I \trianglelefteq S$. So $\sum_{i\in\Lambda} e_iM = \sum_{i\in\Lambda} E_M(e_iS) = E_M(I) \leq^{\oplus} M$. Thus M satisfies the FI-SSSP. $\square$

The following example shows that the converse of Lemma 2.2 is not true, in general.

Example 2.2. Consider the $\mathbb{Z}$-module $\mathbb{Z}_{p^n}$, where p is prime, $n \in \mathbb{N}$ and $n > 1$. $\mathbb{Z}_{p^n}$ satisfies the FI-SSSP as it is indecomposable but $\mathbb{Z}_{p^n}$ is not a quasi-dual Baer $\mathbb{Z}$-module: Let $\phi \in End_{\mathbb{Z}}(\mathbb{Z}_{p^n})$ such that $\phi(x) = px$, $\forall x \in \mathbb{Z}_{p^n}$. The morphism ϕ is not 0 ($p.1 = p \neq 0$ modulo p^n, where $n > 1$); $\operatorname{Im}\phi \triangleleft M$ and since $\mathbb{Z}_{p^n}$ is hollow, $\operatorname{Im}\phi$ cannot be a summand. Therefore $\mathbb{Z}_{p^n}$ is not a quasi-dual Baer $\mathbb{Z}$-module.

Theorem 2.2. *Let M be a module and for all $\phi \in S$, $Im\phi \trianglelefteq M$. Then M is quasi-dual Baer if and only if M has the FI-strong summand sum property (FI-$SSSP$) and $Im\phi \leq^{\oplus} M$, $\forall\phi \in S$.*

Proof. By Lemmas 2.1 and 2.2, M has the FI-strong summand sum property (FI-SSSP) and $\operatorname{Im}\phi \leq^{\oplus} M$, $\forall\phi \in S$. Conversely, let I be any ideal of S. For each $\phi \in I$ we have $\operatorname{Im}\phi \trianglelefteq^{\oplus} M$. Thus $E_M(I) = \sum_{\phi\in I} \operatorname{Im}\phi \leq^{\oplus} M$, by FI-SSSP. Hence M is quasi-dual Baer. $\square$

Theorem 2.3. *Let M be a module and for all $\phi \in S$, $Im\phi \trianglelefteq M$. Then M is quasi-dual Baer if and only if M is dual Baer.*

Proof. It follows from Theorem 2.2 and [10, Theorem 2.1]. $\square$

Following [7], the module M is called *non-cosingular* if for every non-zero module N and every non-zero homomorphism $f : M \to N, \operatorname{Im} f$ is not a small submodule of N. In [9], Keskin-Tütüncü and Tribak introduced the concept of $\mathcal{T}$-non-cosingular modules. According to [9], a module M is called *$\mathcal{T}$-non-cosingular* if, $\forall\phi \in End_R(M)$, $\operatorname{Im}\phi \ll M$ implies that $\phi = 0$. In this paper we introduce the notion of FI-$\mathcal{T}$-non-cosingular modules. A module M is called *FI-$\mathcal{T}$-non-cosingular* if, for any $I \trianglelefteq S$ such that $E_M(I) = eM \oplus D$, where $e^2 = e \in S$ and $D \ll M$, we get that $E_M(I) = eM$. All semisimple modules are FI-$\mathcal{T}$-non-cosingular.

Proposition 2.1. *Let M be an R-module. Then:*

(i) M is $\mathcal{T}$-non-cosingular if and only if, for all $I \leq {}_S S$, $E_M(I) = eM \oplus D$ where $e^2 = e \in S$ and $D \ll M$, implies that $I \cap (1-e)S = 0$.

(ii) M is FI-$\mathcal{T}$-non-cosingular if and only if, for all $I \trianglelefteq S$, $E_M(I) = eM \oplus D$ where $e^2 = e \in S$ and $D \ll M$, implies that $I \cap (1-e)S = 0$.

Proof. (i) Let $I \leq {}_S S$ such that $E_M(I) = eM \oplus D$, where $e^2 = e \in S$ and $D \ll M$. Then $E_M(I \cap (1-e)S) \subseteq E_M(I) \cap (1-e)M = (eM \oplus D) \cap (1-e)M \subseteq (1-e)M \cap (1-e)D$. Since $D \ll M$, $(1-e)D \ll M$. Therefore $(1-e)M \cap (1-e)D \ll M$. Hence $E_M(I \cap (1-e)S) \ll M$. By $\mathcal{T}$-non-cosingularity of M, $I \cap (1-e)S = 0$.

Conversely, let $I \leq {}_S S$ and $E_M(I) \ll M$. Then, by hypothesis, $I \cap S = 0$. Thus $I = 0$.

(ii) Let $I \trianglelefteq S$ such that $E_M(I) = eM \oplus D$, where $e^2 = e \in S$ and $D \ll M$. Since M is FI-$\mathcal{T}$-non-cosingular, $E_M(I) = eM$. Then $E_M(I \cap (1-e)S) \subseteq E_M(I) \cap (1-e)M = eM \cap (1-e)M = 0$. Therefore $I \cap (1-e)S = 0$.

Conversely, let $I \trianglelefteq S$ such that $E_M(I) = eM \oplus D$, where $e^2 = e \in S$ and $D \ll M$. Then, by hypothesis, we have that $I \cap (1-e)S = 0$. Thus $0 = E_M(I \cap (1-e)S) = E_M(I) \cap \sum_{\phi \in (1-e)S} \operatorname{Im} \phi$. Then $E_M(I) \cap \operatorname{Im} \phi = 0$, $\forall \phi \in (1-e)S$. Since $(1-e) \in (1-e)S$, $E_M(I) \cap (1-e)M = 0$. Let $L \leq E_M(I)$ and $D + L = E_M(I) = eM \oplus D$. Then $D + L + (1-e)M = (eM \oplus D) \oplus (1-e)M = M$. Since $D \ll M$, $L + (1-e)M = M$. Then $L + ((1-e)M \cap E_M(I)) = E_M(I)$. Hence $L = E_M(I)$, and so $D \ll E_M(I)$. Since $D \leq^{\oplus} E_M(I)$ and $D \ll E_M(I)$, $D = 0$. Therefore $E_M(I) = eM$. □

Corollary 2.1. *Every $\mathcal{T}$-non-cosingular module is FI-$\mathcal{T}$-non-cosingular.*

Note that every module which is quasi-dual Baer, lifting but not dual Baer has the property that it is FI-$\mathcal{T}$-non-cosingular but not $\mathcal{T}$-non-cosingular.

Recall that a module M is said to be a *$\mathcal{K}$-module* if, $\operatorname{Im} \phi \nleq N$ for all $0 \neq \phi \in S$ implies $N \ll M$ (equivalently, for all $N \leq M$, $D_S(N) = 0$, implies $N \ll M$) [10]. We say that a module M is a *FI-$\mathcal{K}$-module* if, for every $N \trianglelefteq^{\oplus} M$ and $N' \trianglelefteq N$ such that $\operatorname{Im} \phi \nleq N'$, $\forall \phi \in End_R(N)$, we get that $N' \ll N$. All lifting and FI-lifting modules are FI-$\mathcal{K}$-modules (Lemma 2.3).

Proposition 2.2. *Let M be an R-module. Then:*

(i) M is a $\mathcal{K}$-module if and only if, for all $N \leq M$, $E_M(D_S(N)) \leq^{\oplus} M$ implies that $N = E_M(D_S(N)) \oplus D$ such that $D \ll M$.

(ii) M is a FI-$\mathcal{K}$-module if and only if, for all $N \trianglelefteq M$, $E_M(D_S(N)) \leq^{\oplus} M$ implies that $N = E_M(D_S(N)) \oplus D$, where $D \ll M$.

Proof. (i) Let $E_M(D_S(N)) = eM$, for some $e^2 = e \in S$. By Lemma 1.3, $D_S(N) = D_S(eM)$. Since $D_S(eM) \cap D_S((1-e)M \cap N) = 0$ and $D_S((1-e)M \cap N) \subseteq D_S(N) = D_S(eM)$, we obtain that $D_S((1-e)M \cap N) = 0$. As M is a $\mathcal{K}$-module, $(1-e)M \cap N \ll M$. Then $N = E_M(D_S(N)) \oplus ((1-e)M \cap N)$, where $(1-e)M \cap N \ll M$.

Conversely, let $N \leq M$ and $D_S(N) = 0$. Then $E_M(D_S(N)) = 0$. By assume, $N = E_M(D_S(N)) \oplus D$, where $D \ll M$. Then $N = D \ll M$.

(ii) Let $E_M(D_S(N)) = eM$, for some $e^2 = e \in S$. By using the proof of (i), it is enough to show that $N \cap (1-e)M \trianglelefteq M$. As $N \trianglelefteq M$, then $D_S(N) \trianglelefteq S$. So $eM = E_M(D_S(N)) \trianglelefteq M$. Hence $(1-e)M \trianglelefteq M$. Therefore $N \cap (1-e)M \trianglelefteq M$.

Conversely, let $N \trianglelefteq^{\oplus} M$ and $N' \trianglelefteq N$ such that $\operatorname{Im} \phi \nleq N'$, $\forall \phi \in End_R(N) = S'$. Then $D_{S'}(N') = 0$ and so $E_N(D_{S'}(N')) = 0$. By assume, $N' = E_N(D_{S'}(N')) \oplus D$, where $D \ll M$. Then $N' = D \ll M$. Hence $N' \ll N$. □

Corollary 2.2. *Every $\mathcal{K}$-module is a FI-$\mathcal{K}$-module.*

Any module which is dual Baer, FI-lifting but not lifting has the property that it is a FI-$\mathcal{K}$-module but not a $\mathcal{K}$-module. For example, $\mathbb{Q}$ is a FI-$\mathcal{K}$-module but not a $\mathcal{K}$-module.

Lemma 2.3. *Every FI-lifting module M is a FI-$\mathcal{K}$-module.*

Proof. Let $N \trianglelefteq^{\oplus} M$. Then by [8, Proposition 2.10], N is FI-lifting. Take $N' \trianglelefteq N$ such that $\operatorname{Im}\phi \nleq N'$, $\forall \phi \in End_R(N)$. By the FI-lifting property $N' = B \oplus D$ such that $B \leq^{\oplus} N$ and $D \ll N$. Assume $B \neq 0$, hence $N = B \oplus C$ for some R-module C. Then the canonical projection π_2 of N onto B has the property that $\pi_2(N) \subseteq B \subseteq N'$, which is a contradiction. Hence $B = 0$. Then $N' = D \ll N$ and the proof is complete. □

In general, the converse of Lemma 2.3 is not true. The $\mathbb{Z}$-module $\mathbb{Z}$ is a FI-$\mathcal{K}$-module but is not FI-lifting.

Proposition 2.3. *Let M be a quasi-dual Baer FI-$\mathcal{K}$-module. Then M is FI-lifting.*

Proof. Let $N \trianglelefteq M$ and $D_S(N) = eS$ for some $e^2 = e \in S$ (by the quasi-dual Baer property). Hence $E_M(D_S(N)) = eM \leq^{\oplus} M$. By Proposition 2.1, and since M is a FI-$\mathcal{K}$-module, we get that $N = eM \oplus D$, where $D \ll M$. Hence M is FI-lifting. □

Recall that a module M is called *strongly FI-lifting* if, every fully invariant submodule N of M can be written in the form $N = A \oplus D$ where A is a fully invariant direct summand of M and $D \ll M$ [8]. It is clear that every strongly FI-lifting module is FI-lifting.

Remark 2.1. In the proof of Proposition 2.3 we get that $eM \trianglelefteq M$ (since $N \trianglelefteq M$, then $eS = D_S(N) \trianglelefteq S$, hence $eM = E_M(D_S(N)) \trianglelefteq M$), and so we obtain that M is strongly FI-lifting.

Lemma 2.4. *Every quasi-dual Baer module M is FI-$\mathcal{T}$-non-cosigular.*

Proof. Let $I \trianglelefteq S$, with $E_M(I) = eM \oplus D$, where $e^2 = e \in S$, and $D \ll M$. Then by the quasi-dual Baer property, $E_M(I) \leq^{\oplus} M$. Hence $D \leq^{\oplus} M$. Therefore $D = 0$ and so $E_M(I) = eM$. □

The converse of Lemma 2.4 may not be true. For example, the $\mathbb{Z}$-module $\mathbb{Z}$ is FI-$\mathcal{T}$-non-cosigular but is not quasi-dual Baer.

Proposition 2.4. *Let M be a FI-$\mathcal{T}$-non-cosingular FI-lifting module. Then M is quasi-dual Baer.*

Proof. Let $I \trianglelefteq S$. We have that $E_M(I) \trianglelefteq M$, and by the FI-lifting property we get that $E_M(I) =$ $= eM \oplus D$ such that $e^2 = e \in S$ and $D \ll M$. By FI-$\mathcal{T}$-non-cosingularity we have $E_M(I) = eM$. □

Remark 2.2. We note that *FI-$\mathcal{T}$*-non-cosingularity in Proposition 2.4, is not superfluous. For example the $\mathbb{Z}$-module $\mathbb{Z}_{p^n}$, where p is prime, $n \in \mathbb{N}$ and $n > 1$ is FI-lifting but is not a quasi-dual Baer $\mathbb{Z}$-module.

The next result exhibits close connections between quasi-dual Baer modules and FI-lifting modules.

Theorem 2.4. *The following are equivalent for any module M:*
(i) M is a FI-lifting and FI-$\mathcal{T}$-non-cosingular module;
(ii) M is a quasi-dual Baer and FI-$\mathcal{K}$-module.

Proof. By Lemmas 2.3, 2.4 and Propositions 2.3, 2.4. □

Remark 2.3. Theorem 2.4 is a useful source of examples of quasi-dual Baer modules. For example, if R is a right hereditary ring, then every injective module is non-cosingular by [7, Proposition 2.7]. Since every non-cosingular module is FI-$\mathcal{T}$-non-cosingular, every injective FI-lifting module is quasi-dual Baer by Theorem 2.4.

The following Theorem exhibits close links of quasi-dual Baer modules to the strongly FI-lifting modules.

Theorem 2.5. *The following are equivalent for any module M:*
(i) M is a strongly FI-lifting and FI-$\mathcal{T}$-non-cosingular module;
(ii) M is a quasi-dual Baer and FI-$\mathcal{K}$-module.

Proof. By Remark 2.1 and Theorem 2.4. □

Corollary 2.3. *Let M be a FI-$\mathcal{T}$-non-cosingular module. Then M is FI-lifting if and only if M is strongly FI-lifting.*

Proof. By Theorems 2.4 and 2.5. □

We note that, if M is not FI-$\mathcal{T}$-non-cosingular then an FI-lifting module need not be strongly FI-lifting by [8, Remark 3.8].

In general, a direct sum of quasi-dual Baer modules is not quasi-dual Baer, as the following example shows.

Example 2.3. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_p$ and the endomorphism $f : M \to M$ defined by $f(x + \bar{y}) = cy$ with $x \in \mathbb{Z}_{p^\infty},\ y \in \mathbb{Z}$ and c is a non-zero element of $\mathbb{Z}_{p^\infty}$ such that $cp\mathbb{Z} = 0$. It is clear that $\mathrm{Im} f = c\mathbb{Z}$ which is a non-zero submodule of M. Note that $S = End_{\mathbb{Z}}(\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_p) = \begin{pmatrix} End_{\mathbb{Z}}(\mathbb{Z}_{p^\infty}) & Hom_{\mathbb{Z}}(\mathbb{Z}_p,\ \mathbb{Z}_{p^\infty}) \\ 0 & \mathbb{Z}_p \end{pmatrix}$ where $End_{\mathbb{Z}}(\mathbb{Z}_{p^\infty})$ is the ring of p-adic integers. Consider the ideal $I = SfS$ of S, we have $E_M(I) = E_M(SfS) = \sum_{\phi \in SfS} \mathrm{Im}\, \phi$. Since $\mathbb{Z}_{p^\infty}$ is a fully invariant submodule of M, $E_M(SfS)$ is a submodule of $\mathbb{Z}_{p^\infty}$. So $E_M(SfS)$ is a non-zero small submodule of M because $\mathbb{Z}_{p^\infty}$ is hollow. Thus M is not a FI-$\mathcal{T}$-non-cosingular $\mathbb{Z}$-module. By Lemma 2.4, M is not quasi-dual Baer.

Next, we provide some necessary conditions for a (finite) direct sum of quasi-dual Baer modules to be quasi-dual Baer.

Theorem 2.6. *Let M_1 and M_2 be quasi-dual Baer modules. If $\forall x \in M_i,\ \exists \chi \in Hom_R(M_j,\ M_i)$ such that $x \in Im\chi$ $(i \neq j,\ i,\ j = 1,\ 2)$, then $M_1 \oplus M_2$ is a quasi-dual Baer module.*

Proof. Let $S = End_R(M_1 \oplus M_2)$, and let $I \trianglelefteq S$. Then $E_{M_1 \oplus M_2}(I) \trianglelefteq M_1 \oplus M_2$, hence, using Lemma 1.1, $E_{M_1 \oplus M_2}(I) = N_1 \oplus N_2$, where $N_i \trianglelefteq M_i,\ i = 1,\ 2$. As mentioned,

$$S = \begin{pmatrix} S_1 & Hom_R(M_2,\ M_1) \\ Hom_R(M_1,\ M_2) & S_2 \end{pmatrix}.$$

Since $I \trianglelefteq S$ we have the following properties:

$I_1 = \{\phi \in S_1 \mid \phi = \alpha_{11} \text{ with } (\alpha_{ij})_{i,\ j=1,\ 2} \in I\} \trianglelefteq S_1$

$I_2 = \{\phi \in S_2 \mid \phi = \alpha_{22} \text{ with } (\alpha_{ij})_{i,\ j=1,\ 2} \in I\} \trianglelefteq S_2$

We also define $I_{12} = \{\psi \in Hom_R(M_1,\ M_2) \mid \psi = \alpha_{12} \text{ with } (\alpha_{ij})_{i,\ j=1,\ 2} \in I\}$ and $I_{21} = \{\psi \in$ $\in Hom_R(M_2,\ M_1) \mid \psi = \alpha_{21} \text{ with } (\alpha_{ij})_{i,\ j=1,\ 2} \in I\}$.

Let $N'_1 = E_{M_1}(I_1)$. We have that $N_1 = N'_1 + (\sum_{\phi \in I_{21}} \mathrm{Im}\, \phi)$, (or $\sum_{\phi \in I} \mathrm{Im}\, \phi \cap M_1 = \sum_{\theta \in I_1} \mathrm{Im}\theta + \sum_{\psi \in I_{21}} \mathrm{Im}\, \phi$). Since M_1 is a quasi-dual Baer module, $E_{M_1}(I_1) \leq^{\oplus} M_1$. We also have $\sum_{\phi \in I_{21}} \mathrm{Im}\, \phi \leq \sum_{\theta \in I_1} \mathrm{Im}\, \theta = E_{M_1}(I_1) = N'_1$. Then $N_1 = N'_1 \leq^{\oplus} M_1$. □

Theorem 2.7. *Let $\{M_i\}_{i\in\Lambda}$ be a family of quasi-dual Baer modules. If each M_i is isomorphic to a factor of M_j, $\forall i \neq j,\ i,\ j \in \Lambda$, then $M = \bigoplus_{i\in\Lambda} M_i$ is quasi-dual Baer.*

Proof. Let S_i be the endomorphism ring of M_i, $\forall i \in \Lambda$. The endomorphism ring of $M,\ S$, is a ring of matrices, with elements of S_i in the ii-position, and maps $M_j \to M_i$ in the ij-position, $\forall i,\ j \in \Lambda,\ i \neq j$. We need to show that $\forall I \trianglelefteq S,\ E_M(I) \leq^{\oplus} M$. But since $E_M(I) \trianglelefteq M,\ E_M(I) = \bigoplus_{i\in\Lambda} E_M(I) \cap M_i$. We only have to analyze, hence, the column morphism (i.e., matrices) taking M_i into M, for an $i \in \Lambda$. Similar to the proof of Theorem 2.8, we have that the ith column of $I \trianglelefteq S$ has elements from $Hom_R(M_i,\ M_j)$ in the remaining places (call the union of all these sets ω). $E_M(I) \cap M_i = E_{M_i}(I_i) + (\sum_{\phi\in\omega} \operatorname{Im}\phi)$. But $M_i' = E_{M_i}(I_i) \leq^{\oplus} M_i$, since M_i is a quasi-dual Baer module. If we take a $\phi \in \omega$, for example $\phi : M_j \to M_i$ where $i,\ j \in \Lambda$ and $i \neq j$, then $\phi\psi_{ij} \in I_i$, where $\psi_{ij} : M_i \to M_j$ is an epimorphism. Hence $\operatorname{Im}\phi = \operatorname{Im}\phi\psi_{ij} \leq E_{M_i}(I_i)$, then $\sum_{\phi\in\omega} \operatorname{Im}\phi \leq E_{M_i}(I_i)$. Then $E_M(I) \cap M_i = E_{M_i}(I_i) \leq^{\oplus} M_i$. Using this argument for all $i \in \Lambda$, we obtain that $E_M(I) = \bigoplus_{i\in\Lambda} E_{M_i}(I_i) \leq^{\oplus} \bigoplus_{i\in\Lambda} M_i = M$. $\square$

Theorem 2.8. *Let $\{M_i\}_{i\in F}$ be a family of modules. If $M = \oplus_{i\in F} M_i$ is quasi-dual Baer, then $\sum_{\phi\in Hom_R(M_i,\ M_j)} Im\phi \leq^{\oplus} M_j$ for all $i,\ j \in F$ and $i \neq j$.*

Proof. For simplifying notation assume we have M_1 and M_2. We concentrate on $M_1 \oplus M_2$, which is also quasi-dual Baer. Let $N_i = \sum \operatorname{Im}\phi\, \phi \in Hom_R(M_j,\ M_i),\ i,\ j \in \{1,\ 2\}$ and $i \neq j$. We show first that $N_1 \oplus N_2 \trianglelefteq M_1 \oplus M_2$. Take $\alpha \in End_R(M_1 \oplus M_2)$; i.e.

$$\alpha = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$$

where $\phi_{ij} : M_j \to M_i$, for $i,\ j \in \{1,\ 2\}$. Obviously $\phi_{11}(N_1) = \phi_{11}(\sum_{\phi:M_2\to M_1} \operatorname{Im}\phi) = \sum_{\phi:M_2\to M_1} \operatorname{Im}\phi_{11}\phi \subseteq N_1$. Hence $N_1 \trianglelefteq M_1$. Similarly $N_2 \trianglelefteq M_2$.
We have $\phi_{12}(N_2) = \phi_{12}(\sum_{\phi:M_1\to M_2} \operatorname{Im}\phi) = \sum_{\phi:M_1\to M_2} \operatorname{Im}\phi_{12}\phi$. Since $\operatorname{Im}\phi_{12}\phi \leq \operatorname{Im}\phi_{12}$, then $\phi_{12}(N_2) = \sum_{\phi:M_1\to M_2} \operatorname{Im}\phi_{12}\phi \leq \operatorname{Im}\phi_{12} \leq N_1$. Similarly $\phi_{21}(N_1) \leq N_2$. Hence we get that $N_1 \oplus N_2 \trianglelefteq M_1 \oplus M_2$.

Let us now show that $N_1 \oplus N_2 \leq^{\oplus} M_1 \oplus M_2$. Take $D_{S_{12}}(N_1 \oplus N_2)$, where $S_{12} = End_R(M_1 \oplus M_2)$.

Looking at $\alpha \in D_{S_{12}}(N_1 \oplus N_2)$, α a matrix as above, we notice the following: $\phi_{11}(M_1) + \phi_{12}(M_2) \subseteq N_1$. Then $\phi_{11}(M_1) \subseteq N_1$, hence $\phi_{11} \in D_{S_1}(N_1)$. Similarly $\phi_{22} \in D_{S_2}(N_2)$, where $S_1 = End_R(M_1)$ and $S_2 = End_R(M_2)$. At the same time, $\alpha \in End_R(M_1 \oplus M_2)$ such that $\phi_{11} \in D_{S_1}(N_1),\ \phi_{22} \in D_{S_2}(N_2)$ and $\phi_{12},\ \phi_{21}$ are arbitrary in their respective Homs will have the property $\alpha \in D_{S_{12}}(N_1 \oplus N_2)$. Hence

$$D_{S_{12}}(N_1 \oplus N_2) = \begin{pmatrix} D_{S_1}(N_1) & Hom_R(M_2,\ M_1) \\ Hom_R(M_1,\ M_2) & D_{S_2}(N_2) \end{pmatrix}$$

Since $D_{S_{12}}(N_1 \oplus N_2)) \trianglelefteq S_{12},\ E_{M_1\oplus M_2}(D_{S_{12}}(N_1 \oplus N_2)) \trianglelefteq M_1 \oplus M_2$. Hence $E_{M_1\oplus M_2}(D_{S_{12}}(N_1 \oplus N_2)) = N_1' \oplus N_2'$, where $N_1' = E_{M_1\oplus M_2}(D_{S_{12}}(N_1 \oplus N_2)) \cap M_1$ and $N_2' = E_{M_1\oplus M_2}(D_{S_{12}}(N_1 \oplus N_2)) \cap M_2$. It is easily checked that $N_1' = E_{M_1}(D_{S_1}(N_1))\ +\ (\sum_{\psi\in Hom_R(M_2,\ M_1)} \operatorname{Im}\phi)$. Since $E_{M_1}(D_{S_1}(N_1)) \subseteq N_1$ and $\sum_{\psi\in Hom_R(M_2,\ M_1)} \operatorname{Im}\psi = N_1,\ N_1' = N_1$. Similarly for $N_2' = N_2$.

As a result, we obtain that $E_{M_1\oplus M_2}(D_{S_{12}}(N_1 \oplus N_2)) = N_1 \oplus N_2$. In addition to this, since $M_1 \oplus M_2$ is quasi-dual Baer, $E_{M_1\oplus M_2}(D_{S_{12}}(N_1 \oplus N_2)) \leq^{\oplus} M_1 \oplus M_2$. Hence $N_1 \oplus N_2 \leq^{\oplus} M_1 \oplus M_2$. In conclusion (since the indexes were chosen arbitrarily), if M is quasi-dual Baer, then $\sum_{\phi\in Hom_R(M_i,\ M_j)} \operatorname{Im}\phi \leq^{\oplus} M_j$ for $i,\ j \in \{1,\ 2\},\ i \neq j$. $\square$

Recall that a module M is said to have C_2 *condition* if $\forall N \leq M$ with $N \cong M' \leq^{\oplus} M$, we have $N \leq^{\oplus} M$. We say that M have FI-C_2 *condition* if $\forall N \trianglelefteq M$ with $N \cong M' \leq^{\oplus} M$, we have $N \leq^{\oplus} M$.

Proposition 2.5. *Every quasi-dual Baer module has FI-C_2 condition.*

Proof. Let M be a quasi-dual Baer module and N be any fully invariant submodule of M such that $\psi : eM \cong N$ for some $e^2 = e \in End_R(M)$. Set $\phi = \psi e \in End_R(M)$. Then $\operatorname{Im}\phi = \psi eM = N \leq^{\oplus} M$ as M is quasi-dual Baer. □

Recall that a module M is said to have D_2 *condition* if $\forall N \leq M$ with $M/N \cong M' \leq^{\oplus} M$, we have $N \leq^{\oplus} M$. We say that M have *FI-*D_2 *condition* if $\forall N \trianglelefteq M$ with $M/N \cong M' \leq^{\oplus} M$, we have $N \leq^{\oplus} M$.

Proposition 2.6. *Consider the following conditions for an R-module M:*

(i) M is a quasi-dual Baer module with FI-D_2 *condition;*

(ii) M has FI-C_2 *condition and* FI-D_2 *condition and* $\forall\phi \in S$ *with* $\operatorname{Im}\phi \trianglelefteq M$, $\operatorname{Im}\phi$ *is isomorphic to a direct summand of M;*

(iii) $\forall\phi \in S$ *with* $\operatorname{Im}\phi \trianglelefteq M$ *and* $Ker\phi \trianglelefteq M$, *we have* $\operatorname{Im}\phi \leq^{\oplus} M$ *and* $Ker\phi \leq^{\oplus} M$.

Then (i) $\Rightarrow$ *(ii)* $\Rightarrow$ *(iii). If M has FI-SSSP property, then (iii)* $\Rightarrow$ *(i).*

Proof. $(i) \Rightarrow (ii)$ By Proposition 2.5.

$(ii) \Rightarrow (iii)$ Let $\phi \in S$. Since M is quasi-dual Baer, $M/Ker\phi \cong \operatorname{Im}\phi \leq^{\oplus} M$. Thus $Ker\phi \leq^{\oplus} M$ by FI-D_2 condition.

$(iii) \Rightarrow (i)$ It suffices to show that M has FI-D_2 condition. Let N be a fully invariant submodule of M such that $\psi : M/N \cong M' \leq^{\oplus} M$. Set $\phi = \psi\pi \in S$. Then $Ker\phi = Ker\psi\pi = N \leq^{\oplus} M$. □

3. The endomorphism ring of a quasi-dual Baer module

In this section we give a characterization of a quasi-dual Baer module in terms of its endomorphism ring.

Proposition 3.1. *Let M be a quasi-dual Baer module. Then* $S = End_R(M)$ *is a left quasi Baer ring.*

Proof. Let I be an ideal of S. Since M is quasi-dual Baer, $E_M(I) = eM$ where $e^2 = e \in S$. It suffices to show $\ell_S(I) = S(1-e)$. Since for all $\phi \in I$, $\operatorname{Im}\phi \subseteq \sum_{\phi\in I}\operatorname{Im}\phi = E_M(I) = eM$, we have $(1-e)\phi = 0$. Thus $(1-e) \in \ell_S(I)$. Now if $s \in \ell_S(I)$, then $s(\sum_{\phi\in I}\operatorname{Im}\phi) = 0$. So $seM = 0$. Therefore $s = s(1-e) \in S(1-e)$. □

Corollary 3.1. *Let M be a* $\mathcal{T}$*-non-cosingular FI-lifting module. Then S is a left quasi-Baer ring.*

Proof. By Propositions 2.4 and 3.1. □

Our next example shows that the converse of Proposition 3.1 may not be true.

Example 3.1. Let $M = \mathbb{Z}_{\mathbb{Z}}$ be an $\mathbb{Z}$-module. Then $End_R(\mathbb{Z}_{\mathbb{Z}}) \simeq \mathbb{Z}$ is a quasi-Baer ring, but $\mathbb{Z}_{\mathbb{Z}}$ is not a quasi-dual Baer module.

Theorem 3.1. *The following are equivalent for a module M:*

(1) M is a quasi-dual Baer module;

(2) $E_M(I) = r_M(\ell_S(I))$ *for every ideal I of S and S is a left quasi-Baer ring.*

Proof. $(1) \Rightarrow (2)$ Let I be an ideal of S. Since M is quasi-dual Baer, there exists $e^2 = e \in S$ such that $E_M(I) = eM$. Thus $(1-e) \in \ell_S(I)$. Let $m \in r_M(\ell_S(I))$. Then $(1-e)m = 0$, so $m \in eM = E_M(I)$. Therefore $E_M(I) = r_M(\ell_S(I))$. By Proposition 3.1, S is a left quasi-Baer ring.

(2) $\Rightarrow$ (1) Let I be an ideal of S and $\ell_S(I) = Sf$, for some $f^2 = f \in S$. Hence $\forall \phi \in I$, $f\phi = 0$. So $\phi = (1-f)\phi$ and $\phi M \subseteq (1-f)M$. Thus $E_M(I) \subseteq (1-f)M$. But $(1-f)M \subseteq r_M(Sf) = r_M(\ell_S(I)) = E_M(I)$. Therefore M is a quasi-dual Baer module. $\square$

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